

Date :

Collatz conjecture - Veracity or Fallacy..... ?

Abstract : Collatz problem is one the simplest complicated problem of number theory. It says that a seemingly simple program of two steps terminates for all no. s.

Take any, natural no. $m > 0$

$n := m$

repeat

if (n is odd) then $n := 3n+1$,

else $n := n/2$

untill ($n=1$)

Conjecture1. For all positive integers m, the program above terminates.

As no.s are infinite it may be untrue. For a counter example it is a must that in the series any one no. repeats so it may be cyclic from n to n. Whether or not it is true is the post discussed in following pages.

Meaning of Symobls:

x^k - x raised to the power of 'k'

= - is equal to

\cong - is equivalent to

> - is greater than

< - is less than

\Rightarrow - implies

\therefore - Therefore

\because - because

Main body : Collatz problem is one of the most pinching problems of modern mathematics owing to the fact that it is so simple and yet so tough that it resists all solutions. Also known as '3n+1' problem it is a simple algorithm of two simpler steps and asks for whether these two continue for even or not.

To be more precise collatz problem states that the programme of following two steps terminates for all no.s.

Take any, natural no. $m > 0$

$n := m$

repeat

if (n is odd) then $n := 3n+1$,

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Conjecture1. For all positive integers m, the program above terminates.

It just means if n is even half it and if it is odd multiply it by three and add one and go on till $n=1$ i.e smallest natural no.

The complication with this problem is that if it ends for all no. s then how to establish the very fact?

There is infact one way which uses principle of mathematical induction in multiple steps which resolves the problem as true. Hence collatz conjecture is a veracity.

Before proving it let us work out certain basics.

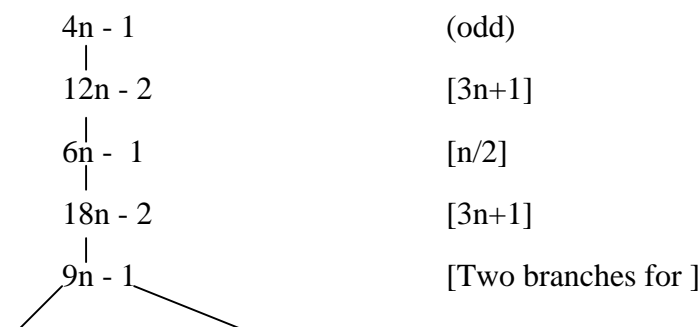
It starts with the fact that for all even integers it holds if it holds for all odd integers. It is due to the fact that eventually all even $2n$ will become $2n/n$ (By step II)

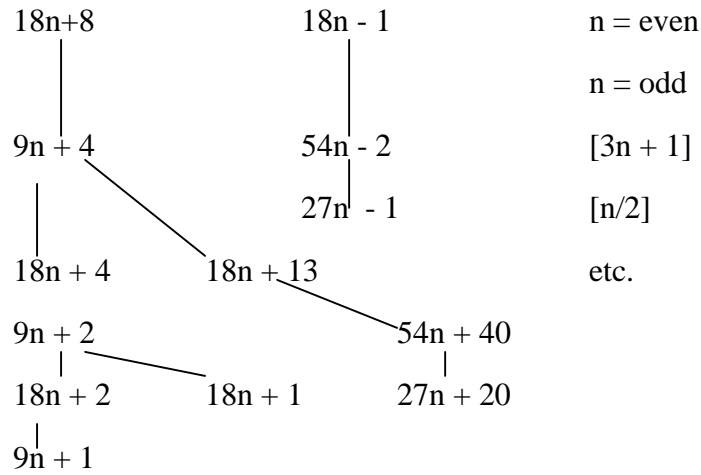
$= n$

Secondly all odd integers are either $4n+1$ or $4n-1$. It would be established that odd integers of the form $4n-1$ reduce to $4n+1$ after multiple application of step (I) & (II), with two exceptions off course.

We would hereforth write the steps in the form of tree where each branch will follow the steps (I) & (II).

A example follows:-





Now we take four cases ie $n = 4k$, $4k \pm 1$ or $4k + 2$.

$\therefore k = 4k + 1$ says

$$\begin{aligned}
 &4n - 1 \\
 &4(4k + 1) - 1 \quad [\text{odd} : 3n + 1] \\
 &12(4k + 1) - 2 \quad [\text{Even} : n/2] \\
 &6(4k + 1) - 1 \\
 &6.4k + 5 \cong 4k + 1
 \end{aligned}$$

$k = 4k + 2$ says

$$\begin{aligned}
 &4n - 1 \\
 &4(4k + 2) - 1 \quad [\text{odd} : 3n + 1] \\
 &12(4k + 1) - 2 \quad [\text{Even} : n/2] \\
 &6(4k + 2) - 1 \quad [\text{odd} : 3n + 1] \\
 &18(4k + 2) - 2 \quad [\text{even} : n/2] \\
 &9(4k + 2) - 1 \\
 &4.9k + 17 \cong 4k + 1
 \end{aligned}$$

$k = 4k$ says

$$\begin{aligned}
 &4.4k - 1 \quad [\text{odd} : 3n + 1] \\
 &12.4k - 2 \quad [\text{even} : n/2] \\
 &6.4k - 1 \quad [\text{odd} : 3n + 1] \\
 &18.4k - 1 \quad [\text{even} : n/2] \\
 &9.4k - 1 \quad [\text{odd} : 3n + 1] \\
 &108k - 2 \quad [\text{even} : n/2]
 \end{aligned}$$

$$\begin{array}{ccc}
 & 54k - 1 & \\
 & \swarrow \quad \searrow & \\
 54(2k + 1) - 1 & & 108k - 1 \quad \text{[one even and other odd]} \\
 | & & \\
 108k + 53 & \cong & 4k + 1
 \end{array}$$

$$\text{Now } 54 \cong 4k + 2$$

Any no. of the form $(4n + 2)k - 1$

reduces to $4n + 1$ if $k = \text{odd}$ & if $k = \text{even}$ then after applying rule (II)

it reduces to

$$3.2(4n + 2).k - 3 + 1 \quad [\text{Even : } n/2]$$

$$\text{or } 3.(4n + 2).k - 1$$

\therefore if $k = \text{odd}$ it reduces to $4k + 1$ form

\therefore if $k = \text{even} = 2^n$ then it is of the form $4k - 1$ otherwise not

Since each $3n + 1$ raises one power of 3 & $3^n.2 - 1 = (2 + 1)^n.2 - 1$

$$\cong 4k + 1 \text{ form.}$$

\therefore only $54.2n - 1$ is remaining form of $4.4k - 1$ which does not reduce to $4k + 1$ form.

\therefore Apart from proving all the no. s of the form $4n + 1$ we got to prove it for $54.2n - 1$ also (A)

It is the first exception.

lastly $n = 4k - 1$ says

$$4(4k - 1) - 1$$

$$12(4k - 1) - 2 \quad [\text{Even : } n/2]$$

$$6(4k - 1) - 1 \quad [\text{odd : } 3n + 1]$$

$$18(4k - 1) - 2 \quad [\text{Even : } n/2]$$

$$9(4k - 1) - 1$$

$$9.4k - 10$$

$$9.2k - 10$$

$$9.2k - 5$$

$$18k - 5 \cong 18k + 18 - 5$$

$$\equiv 18k + 13 \dots\dots\dots (B)$$

It is the second exception.

\therefore $18k + 13$ should also be encountered

apart from all the no. s of the form $4n + 1$.

\therefore If collatz conjecture holds for all the no. s of the form

$$4n + 1 \dots\dots\dots (I)$$

$$54.2^n - 1 \dots\dots\dots (A)$$

$$18n + 13 \dots\dots\dots (B)$$

It is true.

We would start with $\dots\dots\dots (A)$

$$\text{Now } 54.2^n - 1 \quad \equiv \quad 27.2n - 1 \equiv 3^3. 2^n - 1$$

We can easily see that by repeated application of steps (1) & (2) exponent of 3 will rise and that of 2 will fall.

\therefore The end result would be $3^k - 1$

\therefore We have to establish that it holds for all no. s of the form $3^k - 1$

$$\text{Now } 3^k - 1 \equiv 3^n - 1$$

$$\begin{array}{rcl}
 & & 3n - 1 \\
 & \swarrow & \\
 9n - 2 & & \\
 | & & \\
 18n - 2 & \searrow & 18n + 7 \equiv 3n + 1 \\
 | & & \\
 9n - 1 & & \\
 | & & \\
 9n - 1 & & \\
 | & & \\
 27n - 2 & \equiv & 3n + 1
 \end{array}$$

\therefore $n = 3k - 1$ is true if $n = 3n + 1$ is true

Now each $3n + 1$ can be derived from a subseries of $4n + 1$.

That is $4n + 1$

$$\begin{array}{c}
 12n + 4 \\
 | \\
 6n + 2 \\
 | \\
 3n + 1
 \end{array}$$

\Rightarrow Truth of $4n + 1$ implies the truth of $3n + 1$

IIIr results for $18n + 13$ ie $\dots\dots\dots (B)$

∴ Eventually exceptions (A) & (B) also find themselves burried in the truth of $n = 4k + 1$.

∴ their is only one case $n = 4k + 1$ but that isn't that small either.

Till now we have proved that collatz conjecture holds if it holds for all $n = 4k + 1$ type integers. But that isn't half the battle won because one such exception could lead the whole result to a fallacy.

Now comes the vital part of the proof ie. the final case $n = 4k + 1$

The grand finale.....,

The essence of using mathematical induction lies in the fact that if every no. of the form $4k + 1$ terminates and implies that $4k + 5$ also terminates then its true.

The proof is divided in two parts. The first part is the first step of two step induction. Through induction it is established that collatz conjecture holds for all no. s of the $4k + 1$ form except for $3^2 \cdot 2^n + 1$

In Second step it is established that it holds for all no. s of the form $3^2 \cdot 2^n + 1$

Starting with the basic step ie $k = 1$

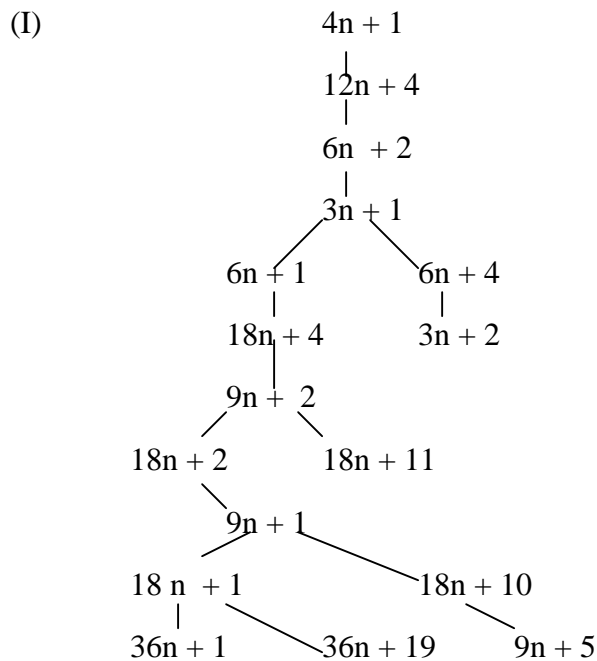
$$\begin{array}{r}
 n = 4 \cdot 1 + 1 = 5 \\
 | \\
 16 \\
 | \\
 8 \\
 | \\
 4 \\
 | \\
 2 \\
 | \\
 1
 \end{array}$$

Now assuming it to be true for $4k + 1$.

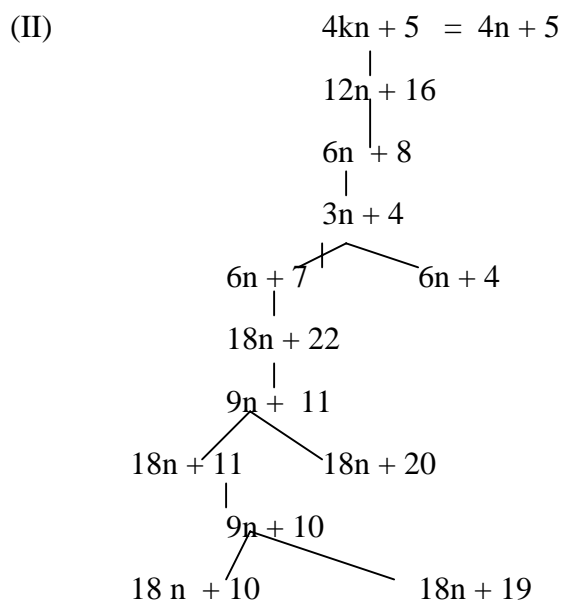
lets prove it to be true for $4k + 5$.

The proof assumes that it a no. in the tree of $4k + 1$ appears in the tree of $4k + 5$ the that part of the tree is assumed to be true. To make it clear that if in the series of $4 \cdot 2^n + 1$ ie 9 any of the 16, 8, 4, 2 occurs then it is true at that step only.

We first make our reference tree for $4n + 1$



Next comes our tree for $4k + 5$.



Comparing the two trees we get that

in (II) the series end at $6n + 4$ as it appears in (I) also.

IIIrd results for $18n + 11$ and $18n + 10$.

\therefore only no. left out is $18n + 19$

ie $18(n + 1) + 1$

Now whenever $n = \text{even}$ it comes in the series of $4n + 1$ as $n = \text{odd}$.

Hence $n + 1 = \text{even}$ or by repeated application $n + 1 \equiv 2^k$.

\therefore Only values to be settled are of the form $3^2 \cdot 2^n + 1$

Hence collatz conjecture is a veracity if it holds for all the no. s of the form $9.2^n + 1$.

Now we would again establish this result through induction.

Hence we will prove it to be true for $P(n) = 9.2^n + 1$, offcourse through induction.

For $n = 1$ $P(1) = 19 \cong 58 \cong 29 \cong 88$

$$\cong 44 \cong 22 \cong 11 \cong 34$$

$$\cong 17 \cong 52 \cong 26 \cong 13 \cong 40$$

$$\cong 20 \cong 10 \cong 5 \cong 16$$

$$\cong 8 \cong 4 \cong 2 \cong 1$$

Now let it hold for

$$P(n) = 9.2^n + 1$$

\Rightarrow Every no. of the form less than

$$9.2^n + 1 \text{ and of the form } 4n + 1$$

Satisfies the collatz conjecture as proved before. (in step one of induction).

$$\therefore P(n+1) = 9.2^{n+1} + 1$$

$$\cong 27.2^{n+1} + 4 \cong 27.2^n + 2$$

$$\cong 27.2^{n-1} + 1$$

\Rightarrow In one step no. is approximately increased 3 times and decreased 4 times

\therefore Net decreased ins around $3/4$ in each step. Hence in three steps it would be $27/64 < 1/2 \Rightarrow$ New no. is less than $1/2$ of $9.2^{n+1} + 1$ and of the form $4k + 1$.

\Rightarrow In tree steps we would get a no. less than $1/2 [9.2^{n+1} + 1]$

ie. $9.2n + 1$.

Hence it holds for $9.2^{n+1} + 1$

Since it holds for each no. less than $9.2^n + 1$

For three steps to happen the min.

value of 'n' is 8 since the no. has to be reduced $1/2^6$ times and the resulting no. should be of the form $4k + 1$.

∴ We have to manually prove it for all the values of $n = 1, 2, \dots, 7$,

For $n = 1$ it has been done.

$$\begin{aligned} n=2, 9 \cdot 4 = 37 &\cong 112 \cong 56 \cong 28 \\ &\cong 14 \cong 7 \cong 22 \cong 11 \cong 34 \\ &\cong 17 \cong 52 \cong 26 \cong 13 \\ &\cong 40 \cong 20 \cong 10 \cong 5 \cong 16 \\ &\cong 8 \cong 4 \cong 2 \cong 1 \end{aligned}$$

$$\begin{aligned} n=3 \quad 9 \cdot 8 + 1 = 73 &\cong 220 \cong 110 \cong 55 \\ &\cong 166 \cong 83 \cong 250 \\ &\cong 125 \cong 376 \cong 188 \cong 94 \\ &\cong 47 \\ 142 &\cong 71 \cong 224 \cong 112 \end{aligned}$$

Now 112 comes in $n = 2$ series hence true.

$$\begin{aligned} n=4 \quad 9 \cdot 16 + 1 &\cong 3 \cdot 9 \cdot 16 + 4 \cong 3 \cdot 9 \cdot 4 + 1 \\ &\cong 3 \cdot 27 \cdot 4 + 4 \cong 81 + 1 = 82 \cong 41 \cong 124 \\ &\cong 62 \cong 31 \cdot 110 \cong 55 \end{aligned}$$

Now 55 comes in previous series.

$$\begin{aligned} n=5 \quad 9 \cdot 2^5 + 1 &\cong 3 \cdot 9 \cdot 2^5 + 4 \cong 27 \cdot 2^3 + 1 \\ &\cong 27 \cdot 3 \cdot 23 + 4 \cong 27 \cdot 3 \cdot 2 + 1 \cong 3 \cdot 81 \cdot 2 + 4 \\ &\cong 3 \cdot 81 + 2 \cong 245 \cong 736 \cong 184 \cong 92 \\ &\cong 46 \cong 23 \cong 70 \cong 35 \cong 106 \cong 53 \\ &\cong 160 \cong 80 \cong 40 \end{aligned}$$

Now 40 comes in the series of $n = 2$

$$\begin{aligned} n=6 \quad 9 \cdot 2^6 + 1 &\cong 3^3 \cdot 2^6 + 4 \cong 3^3 \cdot 2^6 + 4 \cong 3^3 \cdot 2^4 + 1 \cong 3^4 \cdot 2^4 + 4 \\ &\cong 3^4 \cdot 2^2 + 1 \cong 3^5 \cdot 2^2 + 4 \cong 3^5 + 1 \cong 244 \\ &\cong 122 \cong 66 \cong 33 \cong 100 \cong 25 \cong 76 \\ &\cong 19 \cong 58 \cong 29 \cong 88 \cong 44 \cong 22 \cong 11 \end{aligned}$$

Which comes in $n = 2$ series

$$\begin{aligned} n=7 \quad 9 \cdot 2 + 1 &\cong 32 + 3 \cdot 2 + 1 \cong 36 + 2 \cong 731 \\ &\cong 2194 \cong 1097 \cong 3292 \cong 1646 \cong 832 \cong 2470 \end{aligned}$$

$$\begin{aligned}
&\cong 1235 \cong 3706 \cong 1853 \cong 5560 \cong 2780 \\
&\cong 1390 \cong 695 \cong 2086 \cong 1043 \cong 3130 \\
&\cong 1565 \cong 4696 \cong 2348 \cong 1174 \cong 587 \\
&\cong 1762 \cong 881 \cong 2644 \cong 661 \cong 1994 \\
&\cong 997 \cong 2992 \cong 748 \cong 187 \\
&\cong 562 \cong 281 \cong 844 \cong 211 \cong 634 \\
&\cong 317 \cong 1552 \cong 388 \cong 97 \cong 292 \\
&\cong 73 \cong 220 \cong 55
\end{aligned}$$

Which comes in the series of $n=3$.

Henceforth for this seemingly easy looking conjecture seems to hold good.

Hence collatz conjecture is now a theorem.

It does have some useful implications too on our concept of numbers. The most important is that if any no. is subjected to the algorithm of Collatz problem then any of the no. s never repeat because if a no. appearing in the series comes again then it would be a perfect counter example because the chain would run from it to & fro.

Also a word for its various generalizations viz $2n+1$ & $n.r+1$ rules, I guess they may be proved using similar methods (ie of mathematical induction). Given the length of time it took to solve this one I am not going to try any one of them.

Do you have the guts ?

A mathematical challenge!

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